Robotics for IEM (WMIE005)

Final Exam, 29 October 2025

Student Number: .....

1. You have **2 hours** for the exam.
2. Before you start, we recommend that you check all the questions and **plan your time** per question.
3. For every question, **indicate the question number** on the answer sheet.
4. Please write down your (i) student ID number, and (ii) seat number, **very clearly** on **all of the pages that you receive**. Please write your answer **using a (preferably blue) pen, not a pencil**.
5. The exam contains four questions with compulsory sub-questions and an optional sub-question for which bonus points can be earned.
6. In order to receive any bonus points, you **must complete the course evaluation survey** and email a screenshot with your student number to the course coordinator.
7. This is a closed-book exam, but you can use a calculator and **the formula sheet** at the end of this document. Please write down your answer clearly and with proper argumentation/reasoning whenever needed. **Providing only the final answer without proper argumentation is not acceptable**.
8. If you are unclear about a specific problem, you can make your own assumptions. Describe your assumptions at the beginning of your answer.
9. Whenever we think it is appropriate, a follow-up oral exam to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral exam will be based on the questions of this exam and the final grade will be based on the same weighting factor as before where the adjusted grade from the oral exam will be used to replace the final exam grade.
10. If you return the sheets, then your exam will be graded, unless you explicitly write **"do not grade"** on all of the pages. If the exam is graded, it will be registered.

Table 1: This table is for grading purposes only.

	Question 1	Question 2	Question 3	Question 4
Points Obtained	/25	/25	/25	/25
Bonus Points Obtained	/2.5	/2.5	/2.5	/2.5

1. [**25 points**] Answer the following conceptual questions.

   (a) [**8 points**] A robotic manipulator is in an initial state. You are given a set of desired linear and angular velocities of the end-effector frame in the Cartesian space. Describe how the corresponding joint rates can be found. Explain if the corresponding joint rates can always be found. **Hint:** Consider the mathematical relation between the given and the requested variables.

   Solution: Each main part (derivation and discussion) has [50%].

   Let $q \in \mathbb{R}^n$ be the vector of joint variables, and let

   $$V = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

   represent the desired linear and angular velocities of the end-effector in Cartesian space. The **differential kinematic relation** between the joint rates and the end-effector velocities is

   $$V = J(q)\,\dot{q},$$

   where $J(q) \in \mathbb{R}^{6\times n}$ is the manipulator **Jacobian matrix**.

   (i) **Finding the joint rates $\dot{q}$:**

   - For a *square, non-singular Jacobian* ($n = 6$, $\text{rank}(J) = 6$):

     $$\dot{q} = J^{-1}V.$$

   - For a *redundant manipulator* ($n > 6$): the joint rates can be obtained using the pseudoinverse:

     $$\dot{q} = J^{\dagger}V,$$

     where $J^{\dagger} = J^{\top}(JJ^{\top})^{-1}$.

   (ii) **Can the joint rates always be found?**
   Not always. If the Jacobian becomes *rank-deficient* (at a **kinematic singularity**), some Cartesian velocity directions are not achievable or require unbounded joint rates.

   (b) [**6 points**] Consider the problem of dynamics of a manipulator. (i) Describe the Newton-Euler method by indicating whether it is a force-balance method or an energy-based method and the steps that are involved. (ii) Describe the Lagrangian method by indicating whether it is a force-balance method or an energy-based method and the steps that are involved. **Hint:** you do not need to write down any formulas, rather only mention the physical concepts described by the formulas.

   Solution: Each case has [50%]. Mentioning the steps is optional.

   (i) **Newton–Euler method:**
   This is a **force–balance method** based on **Newton's second law** for translation and **Euler's rotational equation** for rotation. It computes the forces and torques acting on each link of the manipulator through two main steps:

   - **Forward recursion:** starting from the base, compute link angular and linear velocities and accelerations.
   - **Backward recursion:** starting from the end-effector, compute joint forces and torques required to produce those accelerations.

   The Newton–Euler formulation focuses on **vector mechanics** (forces, moments, accelerations) and is efficient for recursive numerical implementation.

   (ii) **Lagrangian method:**
   This is an **energy-based method** derived from the **principle of least action** or **energy balance**. It uses the difference between the kinetic and potential energies to describe system dynamics.
   The main conceptual steps are:

- Express the **kinetic energy** T and **potential energy** U in terms of the generalized coordinates and their rates.
- Form the **Lagrangian** $L = T - U$.
- Apply the **Lagrange equations** to obtain the equations of motion for each generalized coordinate.
- Include **dissipative terms** (e.g., damping or friction) if present.

The Lagrangian approach provides a systematic and compact symbolic derivation of the manipulator's equations of motion without explicitly computing all forces and torques.

(c) [**6 points**] Describe the concept of degree of freedom (DOF) required to describe the state of a system. Explain how many degrees of freedom are required to describe the pose of a 3D object in 3D space.

Solution: Each part has [50%].

The **degree of freedom (DOF)** of a system is the number of **independent variables** required to uniquely describe its configuration or state.

For a rigid body in 3D space, fully describing the **pose** (position and orientation) requires:

- 3 DOF to specify its **position** (translation along the x, y, and z axes),
- 3 DOF to specify its **orientation** (rotation about the x, y, and z axes).

Hence, a rigid body in 3D space has a total of **6 degrees of freedom** (3 translational + 3 rotational).

(d) [**5 points**] Describe the concept of inertia tensor by indicating what it represents. **Hint:** you do not need to write down any formulas, rather only mention the physical concepts described by the formulas.

Solution:

The **inertia tensor** describes how the **mass of a rigid body is distributed** relative to a chosen reference point or axis of rotation. It relates the **angular velocity** of the body to its **angular momentum** and determines the body's resistance to changes in rotational motion. Physically, it generalizes the scalar moment of inertia to three dimensions, representing the rotational mass properties of the body.

(e) [**2.5 points - Bonus**] Describe the necessary conditions for the applicability of Pieper's method in the context of inverse manipulator kinematics.

Pieper's method provides a closed-form analytical solution to the inverse kinematics of a manipulator. It is applicable under specific geometric conditions on the joint axes:

Solution:

Pieper's method provides a closed-form analytical solution to the inverse kinematics of a manipulator. It is applicable under specific geometric conditions on the joint axes:

- The manipulator must have **six revolute joints (6R)**.
- The **three adjacent joint axes**—typically the **last three** (axes 4, 5, and 6)—must be either **intersecting at a single point** or **parallel** to each other.

Under these conditions a **closed-form** inverse kinematic solution.

2. [**25 points**] Consider the PRR manipulator shown in Figure 1.

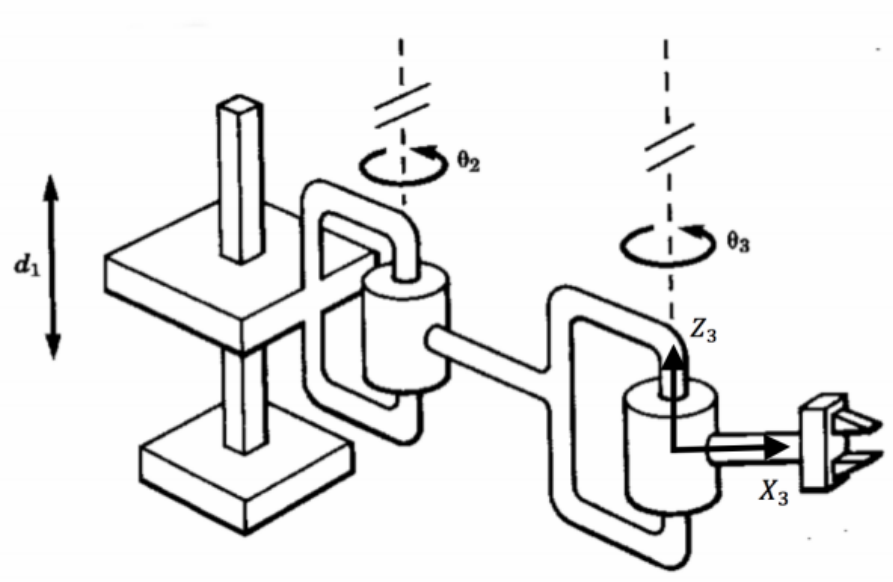


Figure 1: The given PRR manipulator with one prismatic (d_1) and two revolute joints (θ_2 and θ_3). Assume that this image shows a joints configuration where $\theta_2 = \theta_3 = 0$.

(a) [**5 points**] Draw a schematic of this manipulator, with the axes of frames $\{0\}$ through $\{3\}$ labeled. Note that frame $\{3\}$ has been done for you, and your solution should be consistent with the given frame $\{3\}$. Also, include the parameters d_1, a_1, θ_2, a_2 and θ_3 on your schematic. **Hint:** Assume that Figure 1 shows a configuration where $\theta_2 = \theta_3 = 0$.

Solution: Each of the three joint has [20%] and each of the four frames has [10%] of the total points.

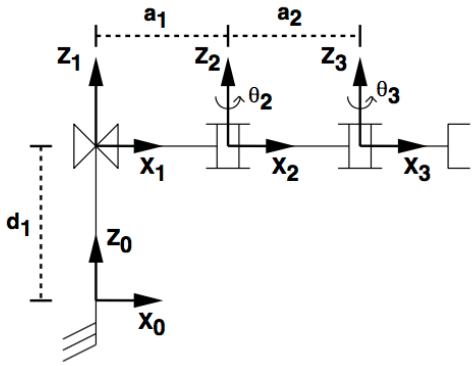


Figure 2: Schematic of manipulator shown in 1

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(b) [**5 points**] Find the DH parameters for this manipulator. You need to assign the interior frames of this manipulator using the frame assignment convention discussed in class.

Solution: Each parameter has [8.33%] of the total points.

Link frame i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	0	d_1
2	0	a_1	θ_2	0
3	0	a_2	θ_3	0

(c) [**5 points**] Derive the forward kinematics, ${}^0_3T$, for this manipulator as a function of ${}^{i-1}_iT$. You need to derive ${}^{i-1}_iT$ matrices. **Hint:** You do not need to perform the matrix multiplications.

Solution: Each ${}^{i-1}_iT$ has [25%]. The functional form of ${}^0_4T$ based on ${}^{i-1}_iT$ has [25%].

$$ {}^0_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & a_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

From above, we have ${}^0_3T = {}^0_1T\,{}^1_2T\,{}^2_3T$ [25%].

(d) [**10 points**] Find the Jacobian matrix (containing both linear and angular parts) in frame 0, namely ${}^0J$, for this manipulator using differentiation of ${}^0\mathbf{P}_{3org}$ if the closed form of the ${}^0_3T$ matrix is given as below. **Hint:** You do not need to perform the matrix multiplications, for the angular part of the Jacobian, you can write the $\hat{Z}_i$ vectors parametrically.

$$ {}^0_3T = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2c_2 + a_1 \\ s_{23} & c_{23} & 0 & a_2s_2 \\ 0 & -c_2 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Solution: The J_v has [50%]. Each element in J_v has [5.6%], each ${}^0\mathbf{Z}_i$ written parametrically has [16.7%]. If you have tried to do matrix multiplications to derive the ${}^0\mathbf{Z}_i$'s, regardless of how far you have gone you get the full points for J_ω of [50%] for showing understanding of the underlying concept. If you have driven the elements of ${}^0\mathbf{Z}_i$'s, you get [25%] extra points on this question.

$$ J_0 = \begin{bmatrix} \frac{\partial {}^0\mathbf{P}_{3org}}{\partial d_1} & \frac{\partial {}^0\mathbf{P}_{3org}}{\partial \theta_2} & \frac{\partial {}^0\mathbf{P}_{3org}}{\partial \theta_3} \\ {}^0\mathbf{Z}_1 & {}^0\mathbf{Z}_2 & {}^0\mathbf{Z}_3 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & -a_2s_2 & 0 \\ 0 & a_2c_2 & 0 \\ 1 & 0 & 0 \\ \epsilon_1{}^0\mathbf{Z}_1 & \epsilon_2{}^0\mathbf{Z}_2 & \epsilon_3{}^0\mathbf{Z}_3 \end{bmatrix} $$

Where we have $\epsilon_1 = 0$ because of prismatic joint and $\epsilon_2 = \epsilon_3 = 1$ because of revolute joint.

(e) [**2.5 points - Bonus**] Write the Jacobian matrix in frame 3, namely ${}^3J$, as a function of ${}^0J$. **Hint:** You do not need to perform the matrix multiplications.

Solution: ${}^3J_v = {}^3_0R\,{}^0J_v$ [50%] and ${}^3J_\omega = {}^3_0R\,{}^0J_\omega$ [50%], you can also combine this in one matrix.

3. [**25 points**] Consider the following RRRR manipulator, shown in Figure 3.

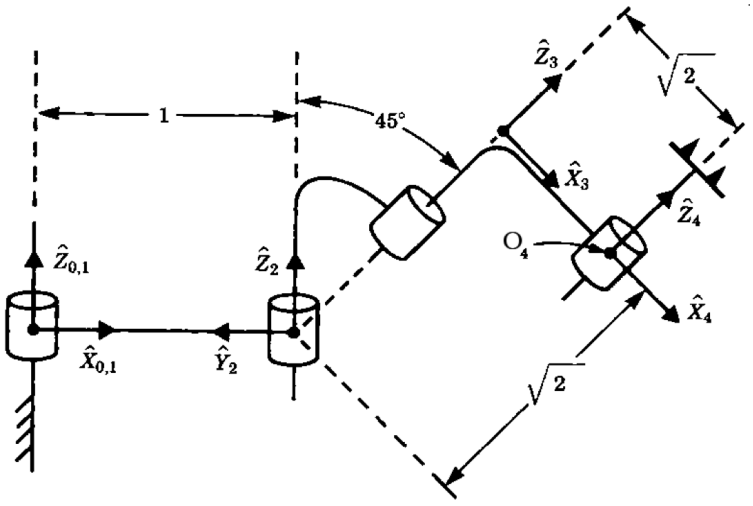


Figure 3: The given RRRR manipulator with four revolute joints (θ_1, θ_2, θ_3, and θ_4). Assume that this image shows a joints configuration where $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$.

(a) [**20 points**] The ${}^{i-1}_{i}T$ matrices are given below. Find the Jacobian matrix (containing both linear J_v and angular J_ω parts) in frame 4, namely ${}^{4}J_\omega$, using the velocity propagation approach. **Hint:** You can write the answer parametrically without simplifying the expressions.

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} -s_2 & -c_2 & 0 & 1 \\ c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} s_3 & c_3 & 0 & 0 \\ -c_3 c_{\pi/4} & s_3 c_{\pi/4} & -s_{\pi/4} & s_{\pi/4}\sqrt{2} \\ 0 & 0 & c_{\pi/4} & c_{\pi/4}\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & \sqrt{2} \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Solution: We discovered that ${}^{2}_{3}T$ had errors in the elements. We graded for the given matrices above. If you used the correct matrices as below, we graded accordingly.

The correct ${}^{2}_{3}T$, denoted ${}^{2}_{3}\hat{T}$ is as below:

$$ {}^{2}_{3}\hat{T} = \begin{bmatrix} s_3 & c_3 & 0 & 0 \\ -c_3 c_{\pi/4} & s_3 c_{\pi/4} & -s_{\pi/4} & -s_{\pi/4}\sqrt{2} \\ -c_3 s_{\pi/4} & s_3 s_{\pi/4} & c_{\pi/4} & c_{\pi/4}\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The solution steps follow. From the given ${}^{i-1}_{i}T$, you must extract ${}^{i-1}_{i}R$ and ${}^{i-1}P_i$. The formulas given in appendix A.7 are as below.

$$
\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \\
{}^{i+1}v_{i+1} &= {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}
\end{aligned}
$$

Solving these iterative formulas requires finding ${}^{i+1}_{i}R$ and ${}^{i}P_i + 1$ for all joints, meaning $i = \{0, 1, 2, 3\}$ and correspondingly ${}^{4}_{3}R = {}^{4}_{3}R^{T}$, ${}^{3}_{2}R = {}^{3}_{2}R^{T}$, ${}^{2}_{1}R = {}^{2}_{1}R^{T}$, and ${}^{1}_{0}R = {}^{1}_{0}R^{T}$ as well as ${}^{3}P_4$, ${}^{2}P_3$, ${}^{1}P_2$, ${}^{0}P_1$. These values can be found as below:

$$
{}^{0}_{1}R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}R = \begin{bmatrix} -s_2 & -c_2 & 0 \\ c_2 & -s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}R = \begin{bmatrix} s_3 & c_3 & 0 \\ -c_3c_{\pi/4} & s_3c_{\pi/4} & -s_{\pi/4} \\ 0 & 0 & c_{\pi/4} \end{bmatrix} = \textit{correction} : {}^{2}_{3}\hat{R} \begin{bmatrix} s_3 & c_3 & 0 \\ -c_3c_{\pi/4} & s_3c_{\pi/4} & -s_{\pi/4} \\ -c_3s_{\pi/4} & s_3s_{\pi/4} & c_{\pi/4} \end{bmatrix}
$$

$$
{}^{3}_{4}R = \begin{bmatrix} c_4 & -s_4 & 0 \\ s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Their transpose is as below, each case has [5%]:

$$
{}^{1}_{0}R = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{1}R = \begin{bmatrix} -s_2 & c_2 & 0 \\ -c_2 & -s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{2}R = \begin{bmatrix} s_3 & -c_3c_{\pi/4} & 0 \\ c_3 & s_3c_{\pi/4} & 0 \\ 0 & -s_{\pi/4} & c_{\pi/4} \end{bmatrix} = \textit{correction} : {}^{3}_{2}\hat{R} \begin{bmatrix} s_3 & -c_3c_{\pi/4} & -c_3s_{\pi/4} \\ c_3 & s_3c_{\pi/4} & s_3s_{\pi/4} \\ 0 & -s_{\pi/4} & c_{\pi/4} \end{bmatrix}
$$

$$
{}^{4}_{3}R = \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

The position vectors are given as below, each case has [5%]:

$$
{}^{0}P_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^{1}P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$^2P_3 = \begin{bmatrix} 0 \\ s_{\pi/4}\sqrt{2} \\ c_{\pi/4}\sqrt{2} \end{bmatrix} \rightarrow \textit{correction}: {}^2\hat{P}_3 = \begin{bmatrix} 0 \\ -s_{\pi/4}\sqrt{2} \\ c_{\pi/4}\sqrt{2} \end{bmatrix}$$

$$^3P_4 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

All joints are revolute, therefore for all joints we have $\dot{d}_i = 0$ for all joints i. Additionally, ${}^i\widehat{Z}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For $i = 0$ we have, each correct placement of vectors and matrices in the equation has [5%]:

$$^1\omega_1 = {}^1_0R\,{}^0\omega_0 + \dot{\theta}_1\,{}^1\widehat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^1v_1 = {}^1_0R\left({}^0v_0 + {}^0\omega_0 \times {}^0P_1\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $i = 1$ we have, each correct placement of vectors and matrices in the equation has [5%]:

$$^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \dot{\theta}_2\,{}^2\widehat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$^2v_2 = {}^2_1R\left({}^1v_1 + {}^1\omega_1 \times {}^1P_2\right) = \begin{bmatrix} c_2\dot{\theta}_1 \\ -s_2\dot{\theta}_1 \\ 0 \end{bmatrix} \rightarrow \quad \omega_1 \times {}^1P_2 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$

For $i = 2$ we have, each correct placement of vectors and matrices in the equation has [5%]:

$$^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \dot{\theta}_3\,{}^3\widehat{Z}_3 = \begin{bmatrix} 0 \\ 0 \\ c_{\pi/4}(\dot{\theta}_1 + \dot{\theta}_2) + \dot{\theta}_3 \end{bmatrix}$$

$$\textit{correction}: {}^3\hat{\omega}_3 = \begin{bmatrix} 0 \\ 0 \\ (-c_3s_{\pi/4} + s_3s_{\pi/4} + c_{\pi/4})(\dot{\theta}_1 + \dot{\theta}_2) + \dot{\theta}_3 \end{bmatrix}$$

$$^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) = \begin{bmatrix} s_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) + c_3s_2\dot{\theta}_1c_{\pi/4} \\ c_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) - s_3s_2\dot{\theta}_1c_{\pi/4} \\ s_2\dot{\theta}_1s_{\pi/4} \end{bmatrix}$$

where $\quad {}^2\omega_2 \times {}^2P_3 = \begin{bmatrix} -(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \\ 0 \end{bmatrix} \quad$ and $\quad {}^2\omega_2 \times {}^2\hat{P}_3 = \begin{bmatrix} (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \\ 0 \end{bmatrix}$

$$\textit{correction}: {}^3\hat{v}_3 = \begin{bmatrix} s_3(c_2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2)) + c_3s_2\dot{\theta}_1c_{\pi/4} \\ c_3(c_2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2)) - s_3s_2\dot{\theta}_1c_{\pi/4} \\ s_2\dot{\theta}_1s_{\pi/4} \end{bmatrix}$$

For $i = 3$ we have, each correct placement of vectors and matrices in the equation has [5%]:

$$
\begin{aligned}
{}^4\omega_4 &= {}^4_3R\,{}^3\omega_3 + \dot{\theta}_4\,{}^4\widehat{Z}_4 \\
&= \begin{bmatrix} 0 \\ 0 \\ c_{\pi/4}(\dot{\theta}_1 + \dot{\theta}_2) + \dot{\theta}_3 + \dot{\theta}_4 \end{bmatrix} \\
\textit{correction}: {}^4\hat{\omega}_4 &= \begin{bmatrix} 0 \\ 0 \\ (-c_3 s_{\pi/4} + s_3 s_{\pi/4} + c_{\pi/4})(\dot{\theta}_1 + \dot{\theta}_2) + \dot{\theta}_3 + \dot{\theta}_4 \end{bmatrix} \\
{}^4v_4 &= {}^4_3R({}^3v_3 + {}^3\omega_3 \times {}^3P_4) \quad \rightarrow \quad {}^3\omega_3 \times {}^3P_4 = \begin{bmatrix} 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \sqrt{2}\dot{\theta}_3 \\ 0 \end{bmatrix} \\
&= {}^4_3R \left(\begin{bmatrix} s_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) + c_3 s_2 \dot{\theta}_1 c_{\pi/4} \\ c_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) - s_3 s_2 \dot{\theta}_1 c_{\pi/4} + (\dot{\theta}_1 + \dot{\theta}_2 + \sqrt{2}\dot{\theta}_3) \\ s_2 \dot{\theta}_1 s_{\pi/4} \end{bmatrix} \right)
\end{aligned}
$$

$$
= \begin{bmatrix} c_4(s_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) + c_3 s_2 \dot{\theta}_1 c_{\pi/4}) + s_4(c_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) - s_3 s_2 \dot{\theta}_1 c_{\pi/4} + (\dot{\theta}_1 + \dot{\theta}_2 + \sqrt{2}\dot{\theta}_3)) \\ -s_4(s_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) + c_3 s_2 \dot{\theta}_1 c_{\pi/4}) + c_4(c_3(c_2\dot{\theta}_1 - (\dot{\theta}_1 + \dot{\theta}_2)) - s_3 s_2 \dot{\theta}_1 c_{\pi/4} + (\dot{\theta}_1 + \dot{\theta}_2 + \sqrt{2}\dot{\theta}_3)) \\ s_2 \dot{\theta}_1 s_{\pi/4} \end{bmatrix}
$$

$$
\textit{correction}: {}^4\hat{v}_4 = {}^4_3R({}^3\hat{v}_3 + {}^3\hat{\omega}_3 \times {}^3P_4) \quad \rightarrow \quad {}^3\hat{\omega}_3 \times {}^3P_4 = \begin{bmatrix} 0 \\ (-c_3 + s_3 + 1)(\dot{\theta}_1 + \dot{\theta}_2) + \sqrt{2}\dot{\theta}_3 \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} c_4(s_3(c_2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2)) + c_3 s_2 \dot{\theta}_1 c_{\pi/4}) + s_4(c_3(c_2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2)) - s_3 s_2 \dot{\theta}_1 c_{\pi/4} + ((-c_3 + s_3 + 1)(\dot{\theta}_1 + \dot{\theta}_2) + \sqrt{2}\dot{\theta}_3)) \\ -s_4(s_3(c_2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2)) + c_3 s_2 \dot{\theta}_1 c_{\pi/4}) + c_4(c_3(c_2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2)) - s_3 s_2 \dot{\theta}_1 c_{\pi/4} + ((-c_3 + s_3 + 1)(\dot{\theta}_1 + \dot{\theta}_2) + \sqrt{2}\dot{\theta}_3)) \\ s_2 \dot{\theta}_1 s_{\pi/4} \end{bmatrix}
$$

Now, we find the Jacobian ${}^4J = \begin{bmatrix} {}^4J_v \\ {}^4J_\omega \end{bmatrix}$ using partial derivatives of ${}^4v_4$ and ${}^4\omega_4$ with respect to θ_i as below; each correct statement has [10%]:

$$
{}^4J_v = \begin{bmatrix} \dfrac{\partial\,{}^4v_4}{\partial\dot{\theta}_1} & \dfrac{\partial\,{}^4v_4}{\partial\dot{\theta}_2} & \dfrac{\partial\,{}^4v_4}{\partial\dot{\theta}_3} & \dfrac{\partial\,{}^4v_4}{\partial\dot{\theta}_4} \end{bmatrix}
$$

$$
{}^4J_\omega = \begin{bmatrix} \dfrac{\partial\,{}^4\omega_4}{\partial\dot{\theta}_1} & \dfrac{\partial\,{}^4\omega_4}{\partial\dot{\theta}_2} & \dfrac{\partial\,{}^4\omega_4}{\partial\dot{\theta}_3} & \dfrac{\partial\,{}^4\omega_4}{\partial\dot{\theta}_4} \end{bmatrix}
$$

(b) [**5 points**] If the joints are set to rotate at a fixed rate of $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 = \pi/4$ per second, find the angular and linear velocities of the end-effector frame after $t = 1s$. **Hint:** You can write the answer parametrically by only specifying the joint variable values.

Solution: Given $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 = \frac{\pi}{4}\,\mathrm{rad/s}$ and $t = 1\,\mathrm{s}$, so $\theta_i(1) = \theta_i(0) + \frac{\pi}{4}$. [50%]

$$\begin{bmatrix} \mathbf{v}_{EE} \\ \boldsymbol{\omega}_{EE} \end{bmatrix} = J(\theta(1)) \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \\ \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix} \quad [50\%]$$

Or equivalently, with $J(\theta) = \begin{bmatrix} J_v(\theta) \\ J_\omega(\theta) \end{bmatrix}$,

$$\mathbf{v}_{EE} = J_v(\theta(1))\,\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{\omega}_{EE} = J_\omega(\theta(1))\,\dot{\boldsymbol{\theta}} \quad [50\%]$$

(c) [**2.5 points - Bonus**] Write the linear Jacobian matrix J_v in frame $\{4\}$ for the position $\mathbf{q} = [0, 90°, -90°, 0]^T$, where $\mathbf{q}$ is the vector of joint variables.

Solution (Bonus 2.5 pts): Given $q = \begin{bmatrix} 0 & 90° & -90° & 0 \end{bmatrix}^\top = \begin{bmatrix} 0 & \frac{\pi}{2} & -\frac{\pi}{2} & 0 \end{bmatrix}^\top$, write the linear Jacobian in $\{4\}$ as a rotation of the base-frame Jacobian:

$$^4J_v(q) \;=\; {}^4_0R(q)\;{}^0J_v(q) \quad [50\%]$$

with ${}^4_0R(q) = \left({}^0_4R(q)\right)^\top$, where we have:

$$^0R_4 \;=\; {}^0R_1\;{}^1R_2\;{}^2R_3\;{}^3R_4. \quad [50\%]$$

4. [**25 points**] Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = -ax_1 + 4x_2 \\ \dot{x}_2 = -bx_1 - x_2^3 \end{cases} \qquad \text{where} \quad a, b \in (0, +\infty)$$

(a) [**12 points**] Find all the equilibrium points of this system and study their stability by means of the Lyapunov function $V(x_1, x_2) \coloneqq x_1^2 + cx_2^2$, with $c \in \mathbb{R}$.

Solution: Finding the equilibrium point has [50%]. Stability analysis has [50%].

The only equilibrium point is in the origin, since

$$\begin{cases} 0 = -ax_1 + 4x_2 \\ 0 = -bx_1 - x_2^3 \end{cases} \implies \begin{cases} x_1 = \frac{4}{a}x_2 \\ bx_1 = -x_2^3 \end{cases} \implies \begin{cases} x_1 = \frac{4}{a}x_2 \\ x_2(\frac{4b}{a} + x_2^2) = 0 \end{cases}$$

Find stability by derivative of Lyapunov function.

The time derivative of the Lyapunov function is equal to:

$$\dot{V}(x_1, x_2) = 2x_1\dot{x}_1 + 2cx_2\dot{x}_2$$

Putting in the system equations gives.

$$\begin{aligned} \dot{V}(x_1, x_2) &= \begin{bmatrix} 2x_1 & 2cx_2 \end{bmatrix} \begin{bmatrix} -ax_1 + 4x_2 \\ -bx_1 - x_2^3 \end{bmatrix} \\ &= -2ax_1^2 + (8 - 2cb)x_1x_2 - 2cx_2^4 \end{aligned}$$

If we choose $c = \frac{4}{b} > 0$, then $\dot{V}(x_1, x_2) < 0 \, \forall x_1, x_2 \neq 0$ and $V(x_1, x_2)$ is positive semi definite. Therefore, the origin is globally asymptotically stable for any value of $a, b \in (0, +\infty)$ and $c = \frac{4}{b} > 0$.

(b) [**8 points**] Describe the concepts of (i) stability and (ii) asymptotic stability in the Lyapanuv sense.

Solution: Each case has [50%], describing required conditions without equations are sufficient.

(i) **Stability in the Lyapunov sense:**
The equilibrium point $x = 0$ is *stable in the sense of Lyapunov* if there exists a function $V(x)$ such that, in a neighborhood of the origin,

$$V(x) > 0 \quad \text{for } x \neq 0, \qquad V(0) = 0$$

(i.e., V is positive definite), and

$$\dot{V}(x) \leq 0$$

(i.e., $\dot{V}$ is negative semidefinite). Then $V(x(t))$ is nonincreasing along trajectories, meaning that trajectories starting sufficiently close to the equilibrium remain close for all $t \geq 0$.

(ii) **Asymptotic stability in the Lyapunov sense:**
The equilibrium point $x = 0$ is *asymptotically stable* if there exists a function $V(x)$ such that

$$V(x) > 0 \text{ for } x \neq 0, \quad V(0) = 0,$$

and

$$\dot{V}(x) < 0 \quad \text{for } x \neq 0$$

(i.e., $\dot{V}$ is negative definite). In this case, $V(x(t))$ strictly decreases along trajectories, and therefore

$$\lim_{t \to \infty} x(t) = 0.$$

(c) [**5 points**] Consider the system of a **hanging** pendulum and consider two cases (i) the joint does not have any damping and (ii) the joint has damping. The joint does not have springiness in both cases. Without writing the equations of motion, describe the stability property of the system around its equilibrium point. **Hinte:** In your description, consider the **hanging** pendulum to be perturbed from its equilibrium state and discuss time evolution of the perturbation.

Solution: Each case has [50%].

Consider the hanging pendulum at its downward equilibrium position.

(i) **No damping (lossless joint):**
A small perturbation from the hanging position leads to *periodic oscillations of constant amplitude* about the equilibrium. The total mechanical energy is conserved, so the perturbation does not decay and the motion remains bounded. Hence, the equilibrium is **Lyapunov stable** but **not asymptotically stable**: trajectories starting sufficiently close remain close for all $t \geq 0$, but they do not converge back to the equilibrium.

(ii) **With damping in the joint:**
A small perturbation produces oscillations whose amplitude *decays over time*. Damping dissipates mechanical energy, so trajectories *spiral back* to the hanging position. Therefore, the equilibrium is **asymptotically stable**: trajectories starting sufficiently close to the equilibrium converge to it as $t \to \infty$.

(d) [**2.5 point - Bonus**] Consider the system of an inverted pendulum that has a rotational joint with both nonzero damping and torsional spring stiffness. Without writing the equations of motion, discuss qualitatively the stability of the system about its upright equilibrium as the spring stiffness varies.

Solution: Discuss the stability of the upright equilibrium for two cases: [50%] for each case.

Consider the inverted pendulum at its upright equilibrium position. The torsional spring provides a restoring torque proportional to the angular displacement from the upright position, while damping dissipates motion energy.

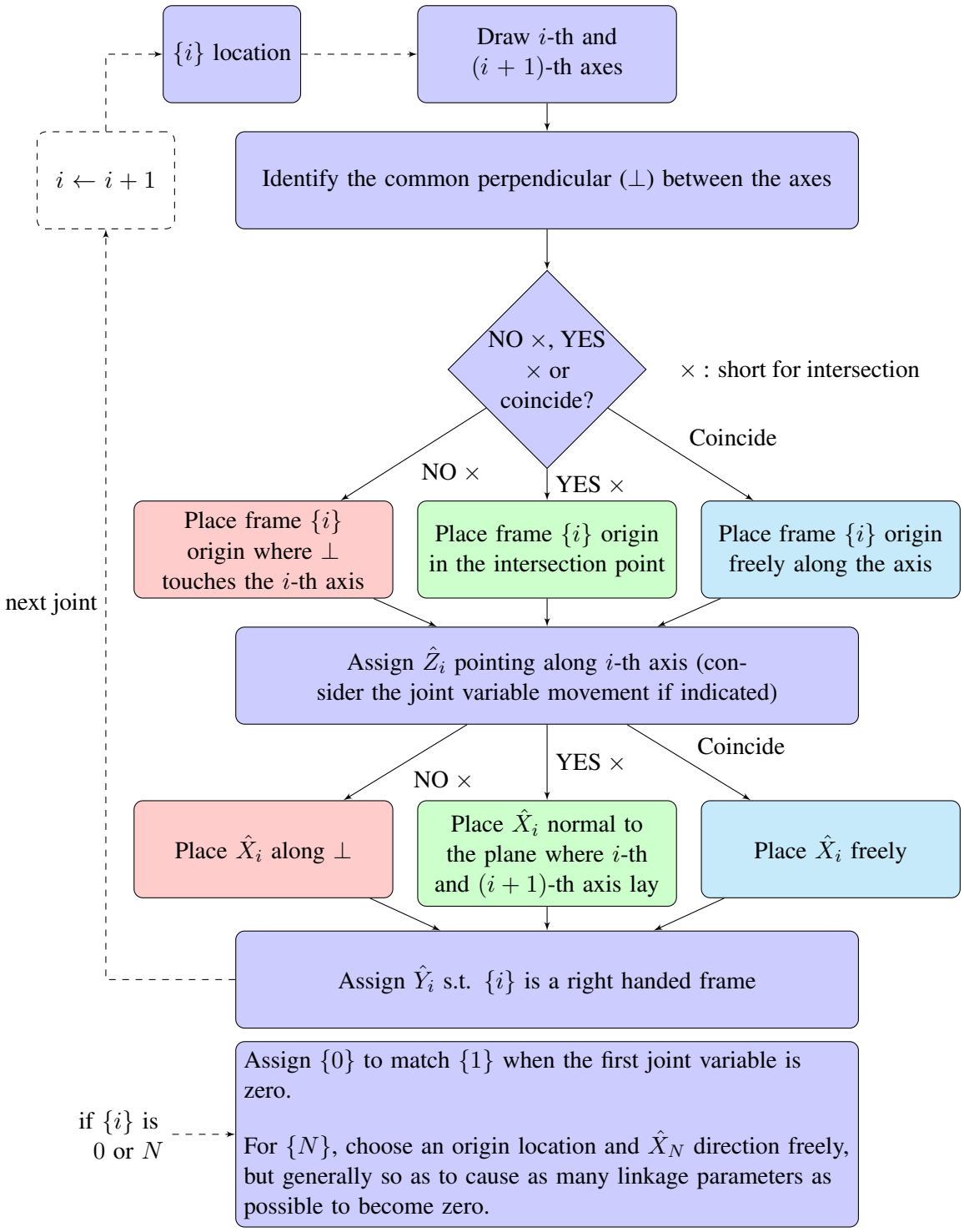
(i) **Spring stiffness smaller than or equal to the critical value ($k \leq k_c$):**
The torsional spring is too weak to overcome the destabilizing torque due to gravity near the upright position. A small perturbation from equilibrium grows with time, and the pendulum tips away from the upright configuration. Even though damping is present, it cannot prevent the motion from diverging if the spring is insufficiently stiff. Hence, the upright equilibrium is **unstable** for $k < k_c$, and at $k = k_c$ it is **Lyapunov stable (not asymptotically stable)**.

(ii) **Spring stiffness larger than the critical value ($k > k_c$):**
The spring torque dominates the gravitational destabilizing effect, and damping removes energy from the motion. Small perturbations from the upright position decay over time, leading the pendulum to return to its equilibrium. Therefore, the upright equilibrium is **locally asymptotically stable** when the spring stiffness exceeds the critical value.

Remark: The critical stiffness k_c represents the value at which the spring's restoring torque exactly balances the gravitational torque near the upright position. Damping ensures convergence only when the net stiffness is stabilizing ($k > k_c$).

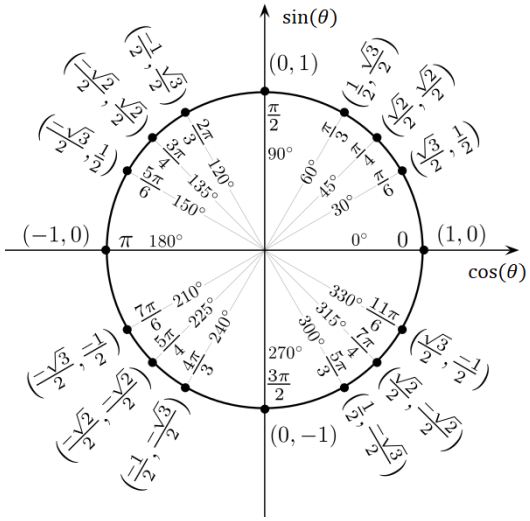
Appendix A - Information Sheet

A.1 Frame assignment



A.2 Trigonometric identities

- $sin(\alpha \pm \beta) = sin(\alpha)cos(\beta) \pm cos(\alpha)sin(\beta)$
- $cos(\alpha \pm \beta) = cos(\alpha)cos(\beta) \mp sin(\alpha)sin(\beta)$
- $sin(-\alpha) = -sin(\alpha)$
- $cos(-\alpha) = cos(\alpha)$
- $\frac{\mathrm{d}}{\mathrm{d}\alpha} sin(\alpha) = cos(\alpha)$
- $\frac{\mathrm{d}}{\mathrm{d}\alpha} cos(\alpha) = -sin(\alpha)$



A.3 Cross product

$$v \times w = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

A.4 Right-hand coordinate system

$\hat{x} = \hat{y} \times \hat{z} \qquad \hat{y} = \hat{z} \times \hat{x} \qquad \hat{z} = \hat{x} \times \hat{y}$

A.5 Basic rotation matrices

$$R_z(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

A.6 General link transformation matrix

$$^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A.7 Angular and linear velocities propagation

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) + \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

A.8 Jacobian matrix

Suppose that the forward kinematics of a robotic manipulator of two degrees of freedom is given by ${}^0P = f(q)$, where $q = [q_1, q_2]^\top$ are the joints variables. Thus, the linear velocities v and angular velocities ω in the workspace and $\dot{q}$ in the joint space are related by the following formulas, where ϵ_i is 0 for prismatic and 1 for revolute joint i.

$$ {}^0v = {}^0J_v(q)\dot{q} \iff {}^0v = \begin{bmatrix} \partial f_1/\partial q_1 & \partial f_1/\partial q_2 \\ \partial f_2/\partial q_1 & \partial f_2/\partial q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} $$

$$ {}^0\omega = {}^0J_\omega(q)\dot{q} \iff {}^0\omega = \begin{bmatrix} \epsilon_1 {}^0\hat{Z}_1 & \epsilon_2 {}^0\hat{Z}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} $$

A.9 Kinetic and potential energy

The total kinetic energy of a rigid robotic system with n links is given by

$$ T(q, \dot{q}) = \sum_{i=1}^{n} \left(\frac{1}{2} m_i v_{ci}^\top v_{ci} + \frac{1}{2} \omega_i^\top I_{ci} \omega_i \right) = \frac{1}{2} \dot{q}^\top M(q) \dot{q} $$

where $m_i, v_{ci}, \omega_i, I_{ci}$ are the mass, linear velocity, angular velocity and inertia (at the center of mass frame $\{c_i\}$) of the i-th link, respectively. The matrix $M(q)$ is the inertia matrix which is symmetric and positive definite. The potential energy is given by

$$ U(q) = \sum_{i=1}^{n} u_i, \text{ where } u_i = -m_i {}^0g^\top {}^0P_{ci} + u_{ref_i} $$

with ${}^0g$ the 3×1 gravity vector, ${}^0P_{ci}$ is the vector locating the center of mass of the i-th link, and u_{ref_i} is a constant chosen so the minimum value of u_i is zero.

A.10 Newton-Euler Iterative approach

Outward iterations:

$$ \begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\, {}^i\omega_i + \dot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} \\ {}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\, {}^i\dot{\omega}_i + {}^{i+1}_{i}R\, {}^i\omega_i \times \dot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} + \ddot{\theta}_{i+1} {}^{i+1}\widehat{Z}_{i+1} \\ {}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R \left({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{i+1} \right) + {}^i\dot{v}_i \right) \\ {}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} \right) + {}^{i+1}\dot{v}_{i+1} \end{aligned} $$

Net forces and torques:

$$ \begin{aligned} {}^{i+1}F_{i+1} &= m_{i+1} {}^{i+1}\dot{v}_{C_{i+1}} \\ {}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1} {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} {}^{i+1}\omega_{i+1} \end{aligned} $$

Inward iterations:

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R^{i+1}f_{i+1} + {}^{i}F_i \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R^{i+1}f_{i+1}
\end{aligned}
$$

Joint torques:

$$\tau_i = {}^{i}n_i^T \, {}^{i}\widehat{Z}_i$$

A.11 Lagrangian approach

Consider the Lagrangian function

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

The equations of motion for a robotic manipulator are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial K}{\partial q}(q, \dot{q}) + \frac{\partial U}{\partial q}(q) = \tau$$

where τ is the vector of joint torques.

A.12 Inertia tensor

The inertia tensor with respect to frame A is calculated as the, where ρ is the density of the object:

$${}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

- $I_{xx} = \iiint (y^2 + z^2)\rho \, dx \, dy \, dz$
- $I_{yy} = \iiint (x^2 + z^2)\rho \, dx \, dy \, dz$
- $I_{zz} = \iiint (x^2 + y^2)\rho \, dx \, dy \, dz$
- $I_{xy} = \iiint xy\rho \, dx \, dy \, dz$
- $I_{xz} = \iiint xz\rho \, dx \, dy \, dz$
- $I_{yz} = \iiint yz\rho \, dx \, dy \, dz$

A.13 Control law partitioning for manipulator control

For the problem of manipulator control, we have the equations of motion as:
$\tau = M(q)\ddot{q} + V(q, \dot{q}) + G(q) + F(q, \dot{q})$
The model-based portion of the control law can be written as:
$\tau = \alpha\tau' + \beta$

$\alpha = M(q)$

$\beta = V(q, \dot{q}) + G(q) + F(q, \dot{q})$
The servo-based portion of the control law can be written as:
$\tau' = \ddot{q}_d + k_v\dot{E} + k_pE$

$E = q - q_d$

A.14 Law of Cosines

- $cos(\alpha) = \frac{b^2+c^2-a^2}{2bc}$
- $cos(\beta) = \frac{a^2+c^2-b^2}{2ac}$
- $cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$

